Common Fixed Point Theorem in Intuitionistic Fuzzy Metric Spaces

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Abstract: In this paper, we prove a common fixed theorem for four mappings under weakly compatible condition in intuitionistic fuzzy metric space.

Key words: Intuitionistic Fuzzy metric space, weakly compatible mappings, common fixed point theorem.

AMS (2010) Subject Classification: 47H10, 54H25

1. Introduction

Atanassove [4] introduced and studied the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets. In 2004, Park [7] defined the notion of intuitionistic fuzzy metric space with the help of continuous $t$-norms and continuous $t$-conorms. Recently, in 2006, Alaca et al.[1] using the idea of Intuitionistic fuzzy sets, defined the notion of intuitionistic fuzzy metric space with the help of continuous $t$-norm and continuous $t$-conorms as a generalization of fuzzy metric space due to Kramosil and Michalek [5]. In this paper, we prove a common fixed theorem for four mappings under weakly compatible condition in intuitionistic fuzzy metric space.

2. Preliminaries:

The concepts of triangular norms ($t$-norms) and triangular conorms ($t$-conorms) are known as the axiomatic skelton that we use are characterization fuzzy intersections
and union respectively. These concepts were originally introduced by Menger [6] in study of statistical metric spaces.

**Definition 2.1.** [9] A binary operation \( * : [0,1] \times [0,1] \rightarrow [0,1] \) is continuous \( t \)-norm if *

\( * \) satisfies the following conditions:

(i) \( * \) is commutative and associative;

(ii) \( * \) is continuous;

(iii) \( a \ast 1 = a \) for all \( a \in [0,1] \);

(iv) \( a \ast b \leq c \ast d \) whenever \( a \leq c \) and \( b \leq d \) for all \( a,b,c,d \in [0,1] \).

**Definition 2.2.** [9] A binary operation \( \diamond : [0,1] \times [0,1] \rightarrow [0,1] \) is continuous \( t \)-conorm if \( \diamond \) satisfies the following conditions:

(i) \( \diamond \) is commutative and associative;

(ii) \( \diamond \) is continuous;

(iii) \( a \diamond 0 = a \) for all \( a \in [0,1] \);

(iv) \( a \diamond b \leq c \diamond d \) whenever \( a \leq c \) and \( b \leq d \) for all \( a,b,c,d \in [0,1] \).

Alaca et al. [1] using the idea of Intuitionistic fuzzy sets, defined the notion of intuitionistic fuzzy metric space with the help of continuous \( t \)-norm and continuous \( t \)-conorms as a generalization of fuzzy metric space due to Kramosil and Michalek [5] as :

**Definition 2.3.** [1] A 5-tuple \( (X,M,N,*,\diamond) \) is said to be an intuitionistic fuzzy metric space if \( X \) is an arbitrary set, \( * \) is a continuous \( t \)-norm, \( \diamond \) is a continuous \( t \)-conorm and \( M, N \) are fuzzy sets on \( X^2 \times [0, \infty) \) satisfying the following conditions:

(i) \( M(x, y, t) + N(x, y, t) \leq 1 \) for all \( x,y \in X \) and \( t > 0 \);

(ii) \( M(x, y, 0) = 0 \) for all \( x,y \in X \);

(iii) \( M(x, y, t) = 1 \) for all \( x,y \in X \) and \( t > 0 \) if and only if \( x = y \);

(iv) \( M(x, y, t) = M(y, x, t) \) for all \( x,y \in X \) and \( t > 0 \);

(v) \( M(x, y, t) \ast M(y, z, s) \leq M(x, z, t + s) \) for all \( x,y,z \in X \) and \( s,t > 0 \);

(vi) for all \( x,y \in X , M(x, y, .) : [0, \infty) \rightarrow [0, 1] \) is left continuous;

(vii) \( \lim_{t \rightarrow \infty} M(x, y, t) = 1 \) for all \( x,y \in X \) and \( t > 0 \).
(viii) \( N(x, y, 0) = 1 \) for all \( x, y \in X \);
(ix) \( N(x, y, t) = 0 \) for all \( x, y \in X \) and \( t > 0 \) if and only if \( x = y \);
(x) \( N(x, y, t) = N(y, x, t) \) for all \( x, y \in X \) and \( t > 0 \);
(xi) \( N(x, y, t) \bowtie N(y, z, s) \geq N(x, z, t + s) \) for all \( x, y, z \in X \) and \( s, t > 0 \);
(xii) for all \( x, y, \in X \), \( N(x, y, t) : [0, \infty) \rightarrow [0, 1] \) is right continuous;
(xiii) \( \lim_{t \to \infty} N(x, y, t) = 0 \) for all \( x, y \in X \).

Then \((M, N)\) is called an intuitionistic fuzzy metric space on \( X \). The functions \( M(x, y, t) \) and \( N(x, y, t) \) denote the degree of nearness and the degree of non-nearness between \( x \) and \( y \) w.r.t. \( t \) respectively.

**Remark 2.1:** Every fuzzy metric space \((X, M, *)\) is an intuitionistic fuzzy metric space of the form \((X, M, 1-M, *, \bowtie)\) such that \(t\)-norm \(*\) and \(t\)-conorm \(\bowtie\) are associated as \( x \bowtie y = 1 - ((1-x) * (1-y)) \) for all \( x, y \in X \).

**Remark 2.2:** In intuitionistic fuzzy metric space \((X, M, N, *, \bowtie)\), \( M(x, y, .) \) is non-decreasing and \( N(x, y, .) \) is non-increasing for all \( x, y \in X \).

Alaca et al. [1] introduced the following notions:

**Definition 2.4:** Let \((X, M, N, *, \bowtie)\) be an intuitionistic fuzzy metric space. Then
(a) a sequence \( \{x_n\} \) in \( X \) is said to be Cauchy sequence if, for all \( t > 0 \) and \( p > 0 \),
\[
\lim_{n \to \infty} M(x_{n+p}, x_n, t) = 1 \quad \text{and} \quad \lim_{n \to \infty} N(x_{n+p}, x_n, t) = 0.
\]
(b) a sequence \( \{x_n\} \) in \( X \) is said to be convergent to a point \( x \in X \) if, for all \( t > 0 \),
\[
\lim_{n \to \infty} M(x_n, x, t) = 1 \quad \text{and} \quad \lim_{n \to \infty} N(x_n, x, t) = 0.
\]

**Definition 2.5:** [1] An intuitionistic fuzzy metric space \((X, M, N, *, \bowtie)\) is said to be complete if and only if every Cauchy sequence in \( X \) is convergent.

In 1996, Jungck [2] introduced the notion of weakly compatible maps as follows:

**Definition 2.6.** Let \((X, M, N, *, \bowtie)\) be an intuitionistic fuzzy metric space. \(f\) and \(g\) be self maps on \( X \). A point \( x \in X \) is called a coincidence point of \( f \) and \( g \) iff \( fx = gx \). In this case, \( w = fx = gx \) is called a point of coincidence of \( f \) and \( g \).
Maps \( f \) and \( g \) are said to be weakly compatible if they commute at coincidence point.
Definition 2.7[8]: A pair of self mappings \((f, g)\) of intuitionistic fuzzy metric space 
\((X, M, N, *, \diamond)\) is said to be reciprocal continuous if
\[
\lim_{n \to \infty} f g x_n = f x, \lim_{n \to \infty} g f x_n = g x
\]
whenever, there exists a sequence \(\{x_n\}\) in \(X\) such that
\[
\lim_{n \to \infty} f x_n = \lim_{n \to \infty} g x_n = x\text{ for some } x \text{ in } X.
\]

Alaca [1] proved the following results:

**Lemma 2.1.** Let \((X, M, N, *, \diamond)\) be an intuitionistic fuzzy metric space and for all \(x, y \in X, t > 0\) and if for a number \(k \in (0,1)\) such that
\[
M(x, y, kt) \geq M(x, y, t) \text{ and } N(x, y, kt) \leq N(x, y, t)
\]
Then \(x = y\).

**Lemma 2.2:** Let \(\{u_n\}\) is a sequence in an intuitionistic fuzzy metric space 
\((X, M, N, *, \diamond)\). If there exists a constant \(k \in (0,1)\) such that
\[
M(u_n, u_{n+1}, kt) \geq M(u_{n-1}, u_n, t) \text{ and } N(u_n, u_{n+1}, kt) \leq N(u_{n-1}, u_n, t) \text{ for all } n = 1, 2, 3, \ldots
\]
Then \(\{u_n\}\) is a Cauchy sequence in \(X\).

3. Main Results

**Theorem 3.1:** Let \(A, B, P\) and \(Q\) be self mappings of a complete intuitionistic fuzzy metric space 
\((X, M, N, *, \diamond)\) satisfying the following:

(3.1) for any \(x, y \in X\), and for all \(t > 0\) there exists \(k \in (0,1)\) such that,
\[
M(Px, Qy, kt) \geq \max \left\{ M(Ax, By, t), \frac{M(Px, Ax, t) + M(Qx, Bx, t)}{2} \right\}
\]
\[
N(Px, Qy, kt) \geq \min \left\{ N(Ax, By, t), \frac{N(Px, Ax, t) + N(Qx, Bx, t)}{2} \right\}
\]

(3.2) \(P(X) \subset B(X)\) and \(Q(X) \subset A(X)\)

(3.3) if one of \(P(X), B(X), Q(X), A(X)\) is complete subset of \(X\) then

(a) \(P\) and \(A\) have a coincidence point

(b) \(Q\) and \(B\) have a coincidence point.

If the pair \((P, A)\) and \((Q, B)\) are weakly compatible then \(A, B, P\) and \(Q\) have a unique common fixed point in \(X\).

**Proof:** As \(P(X) \subset B(X)\) and \(Q(X) \subset A(X)\), so we can define sequences \(\{x_n\}\) and 
\(\{y_n\}\) in \(X\) such that
\[
y_{2n+1} = P x_{2n} = B x_{2n+1}, y_{2n+2} = Q x_{2n+1} = A x_{2n+2}.
\]
By (3.1),
\[ M(Px_{2n}, Qx_{2n+1}, kt) \geq \max \left\{ M(Ax_{2n}, Bx_{2n+1}, t), \frac{M(Px_{2n}, Ax_{2n}, t) + M(Qx_{2n}, Bx_{2n}, t)}{2} \right\} \]
\[ M(y_{2n+1}, y_{2n+2}, kt) \geq \max \left\{ M(y_{2n}, y_{2n+1}, t), \frac{M(y_{2n+1}, y_{2n}, t) + M(y_{2n+2}, y_{2n+1}, t)}{2} \right\} \]
\[ M(y_{2n+1}, y_{2n+2}, kt) \geq \max \left\{ M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}, y_{2n}, t) \right\} \]
\[ M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n}, y_{2n+1}, t) \]

and
\[ N(Px_{2n}, Qx_{2n+1}, kt) \leq \min \left\{ N(Ax_{2n}, Bx_{2n+1}, t), \frac{N(Px_{2n}, Ax_{2n}, t) + N(Qx_{2n}, Bx_{2n}, t)}{2} \right\} \]
\[ N(y_{2n+1}, y_{2n+2}, kt) \leq \min \left\{ N(y_{2n}, y_{2n+1}, t), \frac{N(y_{2n+1}, y_{2n}, t) + N(y_{2n+2}, y_{2n+1}, t)}{2} \right\} \]
\[ M(y_{2n+1}, y_{2n+2}, kt) \leq \min \left\{ N(y_{2n}, y_{2n+1}, t), N(y_{2n+1}, y_{2n}, t) \right\} \]
\[ M(y_{2n+1}, y_{2n+2}, kt) \leq N(y_{2n}, y_{2n+1}, t) \]

Similarly, \( M(y_{2n+2}, y_{2n+3}, kt) \geq M(y_{2n+1}, y_{2n+2}, t) \) and
\[ N(y_{2n+2}, y_{2n+3}, kt) \leq N(y_{2n+1}, y_{2n+2}, t) . \]

Therefore, in general,
\[ M(y_{n}, y_{n+1}, kt) \geq M(y_{n-1}, y_{n}, t) \text{ and } N(y_{n}, y_{n+1}, kt) \leq N(y_{n-1}, y_{n}, t) . \] Hence, by Lemma 2.2, \( \{ y_n \} \) is Cauchy sequence in \( X \). By completeness of \( X \), \( \{ y_n \} \) converges to some point \( z \in X \).

Therefore, subsequence’s \( \{ y_{2n} \}, \{ y_{2n+1} \}, \{ y_{2n+2} \} \) converges to point \( z \in X \) i.e.
\[ \lim_{n \to \infty} Bx_{2n+1} = \lim_{n \to \infty} Px_{2n} = \lim_{n \to \infty} Qx_{2n+1} = \lim_{n \to \infty} Ax_{2n+2} = z . \]

Now, suppose \( A(X) \) is complete, therefore, let \( w \in A^{-1}z \) then \( Aw = z \).

Now, consider,
\[ M(Pw, Qx_{2n+1}, kt) \geq \max \left\{ M(Aw, Bx_{2n+1}, t), \frac{M(Pw, Aw, t) + M(Qw, Bw, t)}{2} \right\} \]
\[ M(Pw, y_{2n+2}, kt) \geq \max \left\{ M(z, y_{2n+1}, t), \frac{M(Pw, z, t) + M(Qw, Bw, t)}{2} \right\} \]

taking \( n \to \infty \), we have
\[ M(P_w, z, k_t) \geq \max \left\{ M(z, z, t), \frac{M(P_w, z, t) + M(Q_w, B_w, t)}{2} \right\} \]
\[ M(P_w, z, k_t) \geq \max \left\{ 1, \frac{M(P_w, z, t) + M(Q_w, B_w, t)}{2} \right\} \]
\[ M(P_w, z, k_t) \geq 1 \]

and

\[ N(P_w, Q_{x_{2n+1}}, k_t) \leq \min \left\{ N(A_w, B_{x_{2n+1}}, t), \frac{N(P_w, A_w, t) + N(Q_w, B_w, t)}{2} \right\} \]
\[ N(P_w, y_{2n+2}, k_t) \leq \min \left\{ N(z, y_{2n+1}, t), \frac{N(P_w, z, t) + N(Q_w, B_w, t)}{2} \right\} \]

taking \( n \to \infty \), we have

\[ N(P_w, z, k_t) \leq \min \left\{ N(z, z, t), \frac{N(P_w, z, t) + N(Q_w, B_w, t)}{2} \right\} \]
\[ N(P_w, z, k_t) \leq \min \left\{ 0, \frac{N(P_w, z, t) + N(Q_w, B_w, t)}{2} \right\} \]
\[ N(P_w, z, k_t) \leq 0 \]

This gives, \( P_w = z = A_w \). Therefore, \( w \) is coincidence point of \( P \) and \( A \).

Since, \( P(X) \subset B(X) \), therefore, \( z = P_w \in P(X) \subset B(X) \), this gives, \( z \in B(X) \), let \( v \in B^{-1}z \) i.e. \( Bv = z \). By (3.1)

\[ M(y_{2n+1}, Q_{v}, k_t) \geq \max \left\{ M(y_{2n+1}, z, t), \frac{M(y_{2n+1}, y_{2n}, t) + M(y_{2n+1}, y_{2n}, t)}{2} \right\} \]

taking \( n \to \infty \),

\[ M(z, Q_{v}, k_t) \geq \max \left\{ M(z, z, t), \frac{M(z, z, t) + M(z, z, t)}{2} \right\} \]
\[ M(z, Q_{v}, k_t) \geq 1 \]

and

\[ N(y_{2n+1}, Q_{v}, k_t) \leq \min \left\{ N(y_{2n}, z, t), \frac{N(y_{2n+1}, y_{2n}, t) + N(y_{2n+1}, y_{2n}, t)}{2} \right\} \]

taking \( n \to \infty \), we have

\[ N(z, Q_{v}, k_t) \leq \min \left\{ N(z, z, t), \frac{N(z, z, t) + N(z, z, t)}{2} \right\} \]
\[ N(z, Q_{v}, k_t) \leq 0 \]
This gives, \( Qv = z = Bv \). So, \( v \) is coincidence point of \( Q \) and \( B \). Since, the pair \( (P, A) \) is weakly compatible, therefore, \( P \) and \( Q \) commute at coincidence point i.e. \( PAw = APw \), this gives, \( Pz = Az \) and as \( (Q, B) \) is weakly compatible, therefore, \( QBv = BQv \) this gives, \( Qz = Bz \).

Now, we will show that \( Pz = z \). By (3.1), we have

\[
M(Pz, Qx_{2n+1}, t) \geq \max \left\{ M(Az, Bx_{2n+1}, t), \frac{M(Pz, Az, t) + M(Qz, Bz, t)}{2} \right\}
\]

and

\[
N(Pz, Qx_{2n+1}, t) \leq \min \left\{ N(Az, Bx_{2n+1}, t), \frac{N(Pz, Az, t) + N(Qz, Bz, t)}{2} \right\}
\]

taking \( n \to \infty \),

\[
M(Pz, z, t) \geq \max \{ M(Az, z, t), 1 \} \geq 1
\]

and

\[
N(Pz, z, t) \leq \min \{ N(Az, z, t), 0 \} \leq 0
\]

This gives, \( Pz = z = Az \).

Similarly, we prove that \( Qz = z \).

By (3.1),

\[
M(Px_{2n}, Qz, t) \geq \max \left\{ M(Ax_{2n}, Bz, t), \frac{M(Px_{2n}, Ax_{2n}, t) + M(Qx_{2n}, Bx_{2n}, t)}{2} \right\}
\]

and

\[
M(y_{2n+1}, Qz, t) \geq \max \left\{ M(y_{2n+1}, Bz, t), \frac{M(y_{2n+1}, y_{2n+1}, t) + M(y_{2n+1}, y_{2n+1}, t)}{2} \right\}
\]

taking \( n \to \infty \),

\[
M(z, Qz, t) \geq \max \left\{ M(z, z, t), \frac{M(z, z, t) + M(z, z, t)}{2} \right\}
\]

and

\[
M(z, Qz, t) \geq \max \{ M(z, Bz, t), 1 \} \geq 1
\]
\[ N(Px_{2n}, Qz, kt) \leq \min \left\{ N(Ax_{2n}, Bz, t), \frac{N(Px_{2n}, Ax_{2n}, t) + N(Qx_{2n}, Bx_{2n}, t)}{2} \right\} \]
\[ N(y_{2n+1}, Qz, kt) \leq \min \left\{ N(y_{2n}, Bz, t), \frac{N(y_{2n+1}, y_{2n+1}, t) + N(y_{2n}, y_{2n}, t)}{2} \right\} \]

taking \( n \to \infty \),
\[ N(z, Qz, kt) \leq \min \left\{ N(z, Bz, t), \frac{N(z, z, t) + N(z, z, t)}{2} \right\} \]
\[ N(z, Qz, kt) \leq \min \{ N(z, Bz, t), 0 \} \]
\[ N(z, Qz, kt) \leq 0 \]

This gives, \( Qz = z = Bz \). Therefore, \( z \) is a common fixed point of \( P, A, Q \) and \( B \).

**For Uniqueness**, let \( w \) be another fixed point of \( P, A, Q \) and \( B \) then by (3.1), we have
\[ M(Pz, Qw, kt) \geq \max \left\{ M(Az, Bw, t), \frac{M(Pz, Az, t) + M(Qz, Bz, t)}{2} \right\} \]
\[ M(z, w, kt) \geq \max \left\{ M(z, w, t), \frac{M(z, z, t) + M(z, z, t)}{2} \right\} \]
\[ M(z, w, kt) \geq 1 \]

and
\[ N(Pz, Qw, kt) \leq \min \left\{ N(Az, Bw, t), \frac{N(Pz, Az, t) + N(Qz, Bz, t)}{2} \right\} \]
\[ N(z, w, kt) \leq \min \left\{ N(z, w, t), \frac{N(z, z, t) + N(z, z, t)}{2} \right\} \]
\[ N(z, w, kt) \leq 0 \]

this gives, \( z = w \). Hence, \( z \) is unique common fixed point of \( P, A, Q \) and \( B \).

By choosing \( P, A, Q \) and \( B \) suitably, one can derive corollaries involving two or more mappings. As a sample, we deduce the following natural result for a pair of self mappings by setting \( P = Q \) in Theorem 3.1:

**Corollary 3.1**: Let \( A, B \) and \( P \) be self mappings of a complete intuitionistic fuzzy metric space \( (X, M, N, *, \emptyset) \) satisfying the following:

(3.4) for any \( x, y \in X \), and for all \( t > 0 \) there exists \( k \in (0,1) \) such that,
\[ M(Px, Py, kt) \geq \max \left\{ M(Ax, By, t), \frac{M(Px, Ax, t) + M(Px, Bx, t)}{2} \right\} \]
\[ N(Px, Py, kt) \leq \min \left\{ N(Ax, By, t), \frac{N(Px, Ax, t) + N(Px, Bx, t)}{2} \right\} \]
(3.5) \( P(X) \subset B(X) \) and \( P(X) \subset A(X) \)

(3.6) if one of \( P(X), B(X), A(X) \) is complete subset of \( X \) then
(a) \( P \) and \( A \) have a coincidence point
(b) \( P \) and \( B \) have a coincidence point.

If the pair \((P, A)\) and \((P, B)\) are weakly compatible then \( A, B \) and \( P \) have a unique common fixed point in \( X \).

By taking \( A = B = I \) (Identity map) in Theorem 3.1, we get

**Corollary 3.2.** : Let \( P \) and \( Q \) be self mappings of a complete intuitionistic fuzzy metric space \((X, M, N, *, \emptyset)\) satisfying the following:

(3.7) for any \( x, y \in X \), and for all \( t > 0 \) there exists \( k \in (0,1) \) such that,

\[
M(Px, Qy, kt) \geq \max \left\{ M(x, y, t), \frac{M(Px, x, t) + M(Qy, x, t)}{2} \right\} \\
N(Px, Qy, kt) \leq \min \left\{ N(x, y, t), \frac{N(Px, x, t) + N(Qy, x, t)}{2} \right\}
\]

(3.8) if one of \( P(X), Q(X) \) is complete subset of \( X \).

If the pair \((P, Q)\) is weakly compatible then \( P \) and \( Q \) have a unique common fixed point in \( X \).

**References:**


